

Friday, September 12, Lecture 7: Linear maps and Matrices

Last time...

- Vector spaces
- $\text{Span}(v_1, \dots, v_n) = \{y = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n : a_i \in \mathbb{R}\}$
- $\{v_1, \dots, v_n\}$ are linearly indep if the only sol to
$$a_1 v_1 + \dots + a_n v_n = 0$$

is $a_1 = a_2 = \dots = a_n = 0$.

- $\{v_1, \dots, v_n\}$ is a basis for V if it's a linearly independent spanning set
- Dimension = # basis vectors

Today

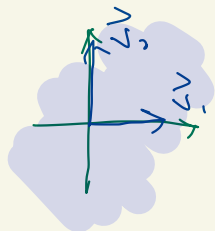
Ex:
 $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ are linearly indep.

Since $a_1 \vec{v}_1 + a_2 \vec{v}_2 = \vec{0} \Leftrightarrow \begin{cases} a_1 + 0a_2 = 0 \\ 0a_1 + 2a_2 = 0 \end{cases}$

$$\text{ref} \left(\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \end{array} \right] \right) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

no free variables and consistent = exactly one solution.

- Geometrically, $\vec{v}_1$ and $\vec{v}_2$ are independent!



Def: Basis

A basis for a vector space V is a linearly independent set of vectors $\{v_1, \dots, v_n\}$ which have $\text{span}(v_1, \dots, v_n) = V$.

Goldilocks Zone:

• Span ensures that we "reach" everywhere in V

Ex: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not span $V = \mathbb{R}^2$, since we cannot write $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ as a linear combo of $\vec{v}_1$, i.e. there is no solution a_1 to

$$a_1 \vec{v}_1 = \vec{v}_2 \quad (\text{no } a \text{ such that } a \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix})$$

- Linear Independence ensures that the list $\{v_1, v_2, \dots, v_n\}$ is not redundant.

Ex:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ has}$$

$\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^2$, but they are not a basis since they are linearly dependent:



: here, $\vec{v}_1 + \frac{1}{2}\vec{v}_2 - \vec{v}_3 = \vec{0}$ is a nonzero solution

$\Rightarrow \vec{v}_3 = \vec{v}_1 + \frac{1}{2}\vec{v}_2$, so you could remove $\vec{v}_3$ without affecting span:
 $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \text{span}(\vec{v}_1, \vec{v}_2)$.

Def: Dimension

The dimension of a vector space V is the number of vectors in a basis. So, if $\{v_1, \dots, v_n\}$ is a basis of V , write

$$\dim(V) = n.$$

Now, a basis provides a special type of coordinate system in a vector space.

Def: Coordinates

Given a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of a vector space V , every vector $w \in V$ may be uniquely written as

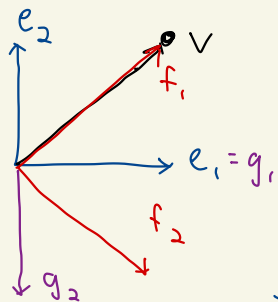
$$w = w_1 v_1 + \dots + w_n v_n$$

we write

$$[w]_{\mathcal{B}} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

and call $[w]_{\mathcal{B}}$ the coordinates of w with respect to the basis $\mathcal{B}$.

Ex: Let $V = \mathbb{R}^2$, and pick the vector $v = (1, 1)$



Define 3 bases:

$$\mathcal{B}_{st} = \{e_1 = (1, 0), e_2 = (0, 1)\}$$

$$\mathcal{B}_{rot} = \{f_1 = (1, 1), f_2 = (1, -1)\}$$

$$\mathcal{B}_{f1} = \{g_1 = e_1, g_2 = -e_2\}$$

Same vector,
different perspectives.

$$v = e_1 + e_2 \Rightarrow [v]_{st} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v = f_1 + 0f_2 \Rightarrow [v]_{rot} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$v = g_1 - g_2 \Rightarrow [v]_{f1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Def: Linear maps.

Let $f: V \rightarrow W$ be a function between vector spaces.

We say f is linear if $\forall v, w \in V, a \in \mathbb{R} \text{ (or } \mathbb{C}),$

$$f(v+w) = f(v) + f(w)$$

$$f(av) = af(v)$$

Ex:

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$f(x) = 3x$ is linear

$$f(x+y) = 3(x+y) = 3x + 3y = f(x) + f(y)$$

$$f(ax) = 3(ax) = a3x = af(x).$$

$f(x) = x^2$ is not linear

$$f(1+1) = (1+1)^2 = 4$$

$$\text{but } f(1) + f(1) = 1^2 + 1^2 = 2.$$

$f(x) = 3x + 5$ is not linear

$$f(1+1) = 3(1+1) + 5 = 6 + 5 = 11$$

$$\text{but } f(1) + f(1) = 3(1) + 5 + 3(1) + 5 = 6 + 10 = 16$$

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$f(x, y, z) = (2y, 3x, 4(z+y))$ is linear

$$\text{Notice: } \underbrace{\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2y \\ 3x \\ 4z + 4y \end{bmatrix}}_{A\vec{x}}$$

Thm

Fix bases $\mathcal{B}_V = \{v_1, \dots, v_n\}$ of V and $\mathcal{B}_W = \{w_1, \dots, w_m\}$ of W . Then

linear maps $A: V \rightarrow W$ $\iff$ $m \times n$ matrices $[A]$, with columns

$$[A] = \begin{bmatrix} | & & | \\ f(v_1) & \dots & f(v_n) \\ | & & | \end{bmatrix}$$

- a vector $u \in V$ is input as coordinates $[u]_V$
- The output vector $Au \in W$ is in coordinates $[Au]_W$
- column vectors are coordinates $[f(v_i)]_W$

How it works:

• Ex: let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\mathcal{B}_{in}$, basis of $\mathbb{R}^3$

let $\tilde{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\tilde{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\mathcal{B}_{out}$, basis of $\mathbb{R}^2$.

Define $A: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $A(x, y, z) = (2x, 3(y+z))$

Let $v \in \mathbb{R}^3$, so

$$v = xe_1 + ye_2 + ze_3 \Rightarrow [v]_{in} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

then

$$[A] = \begin{bmatrix} | & | & | \\ [A(e_1)]_{out} & [A(e_2)]_{out} & [A(e_3)]_{out} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 3 \end{bmatrix}$$

$\uparrow$
 $A(e_1) = 2\tilde{e}_1$, so coordinates are $[A(e_1)]_{out} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

check:

$$[A] \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{in} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{in} = \begin{bmatrix} 2x \\ 3y+3z \end{bmatrix}_{out}$$

The right perspective can greatly clarify the action of a map.

Ex:

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be $T(x,y) = (x+y, x+y)$

Take the standard basis $\mathcal{B}_{\text{st}} = \{e_1 = (1,0), e_2 = (0,1)\}$.

Then

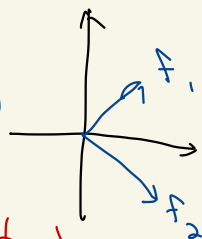
$$[T]_{\text{st}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \leftarrow \text{not diagonal...}$$

But now, choose new basis $\mathcal{B}_{\text{new}} = \{f_1 = (1,1), f_2 = (1,-1)\}$

Notice that

$$Tf_1 = 2f_1 \leftarrow f_1 \text{ and } f_2 \text{ are eigenvectors!}$$

$$Tf_2 = 0$$



New coordinates: any $v = v_1 f_1 + v_2 f_2$ is

$$[v]_{\text{new}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

so

$$[f_1]_{\text{new}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [f_2]_{\text{new}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow [T]_{\text{new}} = \begin{bmatrix} [T(f_1)]_{\text{new}} & [T(f_2)]_{\text{new}} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\leftarrow$... but it's diagonal in this basis!

Moral: the linear operator $T: V \rightarrow V$ is represented by a diagonal matrix when V has a basis of eigenvectors of T .

We really like diagonal matrices, they're the best.

Def: Diagonalizable matrices

An $n \times n$ matrix M is called diagonalizable if there is an invertible matrix S such that

$$M = SDS^{-1}$$

where D is diagonal.

• Of course, by replacing S with R^{-1} , we could also say

$$M = R^{-1}DR.$$

• We know that given a basis $\mathcal{B}$ of a vector space V with $n = \dim(V)$,

$$\begin{array}{ccc} \text{linear operators} & \longleftrightarrow & n \times n \text{ matrices} \\ T: V \rightarrow V & & [T]_{\mathcal{B}} \end{array}$$

so what does this mean for linear operators?

Let $T: V \rightarrow V$ be a linear operator.

Correspondence:

There is a basis of V
consisting of eigenvectors
 $v_1, \dots, v_n$ of T .



The matrix $[T]$ is
diagonalizable.

Why? • Suppose there is a basis of eigenvectors $\mathcal{B}_{\text{eig}} = \{v_1, \dots, v_n\}$ of T ,

so

$$T v_i = \lambda_i v_i \quad \text{for all } i = 1, \dots, n.$$

The coordinates of the eigenvector v_i in $\mathcal{B}_{\text{eig}}$ are easily found:

$$v_i = 0v_1 + \dots + 0v_{i-1} + v_i + 0v_{i+1} + \dots + 0v_n \Rightarrow [v_i]_{\text{eig}} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th slot.}$$

so in this basis,

$$[T]_{\text{eig}} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

check:

$$\begin{aligned} [T]_{\text{eig}} [v_i]_{\text{eig}} &= \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \\ &= \lambda_i [v_i]_{\text{eig}} \end{aligned}$$

• Likewise, if $\mathcal{B}$ is some basis and $[T]_{\mathcal{B}}$ is diagonalizable,

$$[T]_{\mathcal{B}} = S [T]_{\text{eig}} S^{-1}$$

← change-of-basis matrix.